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Free boson formulation of boundary states in W_3 minimal models and the critical Potts model

Alexandre F. Caldeira

Department of Physics, University of Oxford
Theoretical Physics
1 Keble Road, Oxford OX1 3NP, UK
E-mail: caldeira@thphys.ox.ac.uk

Shinsuke Kawai

Helsinki Institute of Physics,
P.O.Box 64, FIN-00014 University of Helsinki, Finland
E-mail: shinsuke.kawai@helsinki.fi

John F. Wheater

Department of Physics, University of Oxford
Theoretical Physics
1 Keble Road, Oxford OX1 3NP, UK
E-mail: j.wheater@physics.ox.ac.uk

Abstract: We develop a Coulomb gas formalism for boundary conformal field theory having a W symmetry and illustrate its operation using the three state Potts model. We find that there are free-field representations for six W conserving boundary states, which yield the fixed and mixed physical boundary conditions, and two W violating boundary states which yield the free and new boundary conditions. Other W violating boundary states can be constructed but they decouple from the rest of the theory. Thus we have a complete free-field realization of the known boundary states of the three state Potts model. We then use the formalism to calculate boundary correlation functions in various cases. We find that the conformal blocks arising when the two point function of $\phi_{2,3}$ is calculated in the presence of free and new boundary conditions are indeed the last two solutions of the sixth order differential equation generated by the singular vector.

Keywords: Conformal and W Symmetry, Boundary Quantum Field Theory.

Dedicated to the memory of Ian I. Kogan, 1958-2003

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Contents

1. Introduction

The Coulomb gas formalism [1, 2] provides a powerful method for calculating correlation functions and conformal blocks in minimal rational conformal field theories (CFTs) without having to solve complicated differential equations. Boundary CFTs have been of great interest since Cardy's famous paper [3] and recently it has been shown that free-field representations may be extended from bulk CFTs to systems with boundary(ies) [4, 5] in the case of the Virasoro diagonal minimal models. The boundary states appear as coherent states in the free-field formalism. As in the bulk case these methods again provide a mechanism for calculating correlation functions, this time in the presence of a boundary.

The three state Potts model is of particular interest because the conformal field theory describing its critical point is the simplest in which there is a higher dimensional chiral operator of integer dimension. The resulting extended symmetry, generated by a W_3 algebra, has important consequences. In particular the theory is diagonal in the bigger W_3 algebra but not in the smaller Virasoro algebra. Such theories can be represented by a multi-component Coulomb gas formalism [6]. A great deal is known about the boundary states in this model. There are six states originally found by Cardy in which the W current is conserved at the boundary [7] and which correspond to fixed and mixed boundary

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conditions in the spin system. In addition there are known to be two more states in which the W current is not conserved. One of them is the state corresponding to free boundary conditions on the spin system, which was of course expected to exist, and the other is the so-called new boundary condition; these were found in [8]. Subsequently it was shown in [9] that the set of eight states is in some sense complete and in [10] it was suggested that this completeness amounts to the set of boundary conditions for which the model is an integrable system. Clearly none of these boundary states can correspond to boundary conditions on the spin system which are not conformally invariant.

The purpose of this paper is to show how the multi-component Coulomb gas formalism may be used to provide a free-field description of the boundary states in theories with a W algebra using the three state Potts model to illustrate the method. We start in section 2 by briefly reviewing those aspects of the standard formalism that we need. In section 3 we record in detail the representations of the primary fields for the Potts model and in section 4 explain how to construct the free-field representations for Cardy's boundary states. Section 5 contains some example calculations of correlation functions in the presence of boundaries; these act as a useful cross-check that the states we have constructed are correct.

Finally in section 6 we discuss some open issues and some useful results are recorded in the appendices.

2. Free-field construction of CFTs with W algebra

The usual Coulomb gas formalism [1, 2] can be extended to CFTs with a larger symmetry than the Virasoro algebra by introducing a multiple component scalar field [6] $\Phi^j(z,\bar{z})$, $j=1,\ldots,r$, which is a vector in the root space of a finite dimensional Lie Algebra $\mathcal{A}$ of rank r. In practice we will be considering the simplest case in this paper so we will confine ourselves to the algebra A_r from the outset. Let us first fix some notation. The simple roots will be denoted $e_j, j=1,\ldots,r$, and the corresponding dual weights $\omega_j, j=1,\ldots,r$. We will use "$\cdot$" to denote multiplication of vectors and matrices in the root space. So the scalar product of two vectors u and v in the root space will be written $u\cdot v$, the product of two matrices $m_1\cdot m_2$ and so on. The simple roots and dual weights then satisfy

$$e_j\cdot e_j = 2\,, \qquad e_j\cdot e_{j+1} = -1\,, \qquad e_j\cdot\omega_i = \delta_{i,j}\,. \tag{2.1}$$

The Weyl vector ρ is defined as

$$\rho = \sum_{j=1}^{r}\omega_j \tag{2.2}$$

and the fundamental weights $h_K, K=1,\ldots,r+1$, satisfy

$$\begin{aligned} h_1 &= \omega_1\,, \\ h_K - h_{K+1} &= e_K\,. \end{aligned} \tag{2.3}$$

We denote the Weyl group of $\mathcal{A}$ by $\mathcal{W}$, an element of it by w, and let $\varepsilon_w = \det w$; it is convenient also to define the shifted Weyl transformation by

$$w^*(\gamma) = w(\gamma+\rho)-\rho. \tag{2.4}$$

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In the particular case of A_2 we have the useful relations

$$\begin{aligned} e_1 &= 2\omega_1 - \omega_2\,, \\ e_2 &= 2\omega_2 - \omega_1\,, \\ h_2 &= \omega_2 - \omega_1\,, \\ h_3 &= -\omega_2\,. \end{aligned} \tag{2.5}$$

The Weyl group is of order 6 and is generated by w_1 (reflections in e_1) and w_2 (reflections in e_2)

$$\mathcal{W} = \{w_0 = I, w_1, w_2, w_3 = w_1 w_2 w_1 \equiv w_\rho, w_4 = w_2 w_1, w_5 = w_1 w_2\}\,. \tag{2.6}$$

The action for Φ takes the usual form

$$\mathcal{S}[\Phi] = \frac{1}{8\pi}\int d^2z\sqrt{g}\,\left(\partial_\mu\Phi\cdot\partial^\mu\Phi + 4i\alpha_0\rho\cdot\Phi R\right), \tag{2.7}$$

where R is the scalar curvature, g the metric, and α_0 a constant. We now split Φ into a holomorphic component $\phi(z)$ and an anti-holomorphic component $\bar{\phi}(\bar{z})$. The field ϕ has mode expansion

$$\phi^j(z) = \phi_0^j - i a_0^j \ln z + i\sum_{n\neq 0}\frac{a_n^j}{n}z^{-n}\,, \tag{2.8}$$

and similarly for $\bar{\phi}$. Canonical quantization gives the usual commutation relations

$$[a_m^j, a_n^l] = m\delta^{jl}\delta_{m+n,0}\,, \tag{2.9}$$

$$[\phi_0^j, a_0^l] = i\delta^{jl}\,. \tag{2.10}$$

Variation of the action with respect to the metric yields the energy-momentum tensor

$$T(z) = -2\pi T_{zz} = -\frac{1}{2}:\partial\phi\cdot\partial\phi: + 2\,i\,\alpha_0\,\rho\cdot\partial^2\phi\,, \tag{2.11}$$

which has the usual expansion

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}\,, \tag{2.12}$$

where the operators

$$L_n = \frac{1}{2}\sum_{m\in\mathbb{Z}} :a_m\cdot a_{n-m}: - 2\alpha_0(n+1)\rho\cdot a_n \tag{2.13}$$

obey the Virasoro algebra with central charge

$$c = r - 48\,\alpha_0^2\rho^2\,. \tag{2.14}$$

Fock spaces $\mathcal{F}_\alpha$ are labeled by a vacuum $|\alpha\rangle$, which is an eigenvector of the a_0^j operator, and annihilated by the positive modes

$$\begin{aligned} a_0^j|\alpha\rangle &= \alpha^j|\alpha\rangle\,, \\ a_n^j|\alpha\rangle &= 0\,, \qquad n > 0\,. \end{aligned} \tag{2.15}$$

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The Fock space is formed by applying the creation operators to the vacuum,

$$a^{j_1}_{-n_1} a^{j_2}_{-n_2} \dots a^{j_p}_{-n_p} |\alpha\rangle \,, \tag{2.16}$$

and different Fock spaces are related by

$$e^{i\beta\cdot\phi_0}|\alpha\rangle = |\beta+\alpha\rangle \,, \tag{2.17}$$

as can be checked using the commutation relations (2.10). The physical Hilbert space of the theory is much smaller than the direct sum of the Fock spaces and is given by the cohomology of a nilpotent BRST operator [11].

The chiral vertex operators for this theory are defined by generalizing the vertex operators of the U(1) Coulomb gas formalism,

$$V_\alpha(z) =: e^{i\alpha\cdot\phi(z)} : \,. \tag{2.18}$$

It is straightforward to check, by computing the OPE $T(z)V_\alpha(z')$, that vertex operators are primary fields with conformal dimensions given by

$$h(\alpha) = \frac{1}{2}\alpha\cdot(\alpha - 4\alpha_0\rho) \,. \tag{2.19}$$

Note that

$$h(\alpha) = h(4\alpha_0\rho - \alpha) \,. \tag{2.20}$$

This generalized free-field construction contains an extra symmetry which corresponds to the presence of the higher dimensional conserved chiral primary fields found in theories with a W symmetry (see [12, 13] for reviews). The new chiral fields can be generated systematically in the following way. First define the generating functional $\mathcal{D}_N$, where $N = r+1$, by

$$(2i\alpha_0)^N \mathcal{D}_N =: \prod_{K=1}^{N} (2i\alpha_0\partial_z + h_K\cdot\partial\phi(z)) : \,. \tag{2.21}$$

This can be evaluated to get

$$\mathcal{D}_N = \partial^N + \sum_{K=1}^{N} (2i\alpha_0)^{-K} u_K[\phi(z)]\, \partial^{N-K}, \tag{2.22}$$

where it turns out that $u_1 = 0$ and $u_2 = T(z)$ and the higher W currents are given by

$$\begin{aligned} W_K = u_K + b\partial u_{K-1} &+ b'\partial^2 u_{K-2} + \cdots \\ &+ b'' u_2 u_{K-2} + \cdots \\ &\vdots \qquad\qquad\qquad , \end{aligned} \tag{2.23}$$

where b etc are constants. For our purposes in this paper it is sufficient to know the generating functional $\mathcal{D}_N$ and the zero-mode of $W_3(z)$ for A_2 which is,

$$
\begin{aligned}
W_0 = ix^{-1}\Bigg(& \sum_{l+m+n=0} h_1 \cdot a_l \, h_2 \cdot a_m \, h_3 \cdot a_n + \\
& + 2\alpha_0 \sum_m [(1-m) h_3 \cdot a_m \, h_1 \cdot a_{-m} + 2h_2 \cdot a_m \, h_1 \cdot a_{-m}] + \\
& + 8\alpha_0^2 \, h_1 \cdot a_0 + 2\alpha_0 L_0 \Bigg) ,
\end{aligned} \tag{2.24}
$$

where $x = \sqrt{(22+5c)/48}$.

Screening operators Q_α are defined by analogy with the one component Coulomb gas formalism as

$$Q_\alpha = \oint dz : e^{i\alpha \cdot \phi(z)} : , \tag{2.25}$$

where now α is any vector living on the $r-1$ dimensional manifold $h(\alpha) = 1$. Since for any such α

$$T(z) V_\alpha(w) = \partial_w(\ldots) + \text{regular terms} , \tag{2.26}$$

the Q_α commute automatically with the Virasoro algebra,

$$[L_n, Q_\alpha] = 0 . \tag{2.27}$$

However in general the Q_α do not commute with the W algebra and it can be shown (see appendix A) that the only ones which do are (in the particular case of A_2)

$$
\begin{aligned}
Q_\pm^{(1)} &= \oint dz : e^{i\alpha_\pm e_1 \cdot \phi(z)} : , \\
Q_\pm^{(2)} &= \oint dz : e^{i\alpha_\pm e_2 \cdot \phi(z)} : .
\end{aligned} \tag{2.28}
$$

3. Free-field approach to the three state Potts model

The three state Potts model is a minimal model that is not diagonal in the "small" Virasoro algebra but becomes diagonal in the larger W_3 algebra. To find its representation in terms of the A_2 Coulomb gas we have to find irreducible highest weight representations for which the Verma modules are completely degenerate, that is to say have as many singular vectors as possible. This can be done by considering a theory, $W_3(p, p')$, parametrized by relative primes p and $p\prime$ where

$$
\begin{aligned}
2\alpha_0 &= \frac{p' - p}{\sqrt{pp\prime}} , \\
\alpha_+ &= \frac{p\prime}{\sqrt{pp\prime}} , \\
\alpha_- &= -\frac{p}{\sqrt{pp\prime}} ,
\end{aligned} \tag{3.1}
$$

for which the central charge is

$$c^{(3)}_{(p,p')} = 2 - 24\frac{(p'-p)^2}{pp'} . \tag{3.2}$$

Choosing $p = 4$, $p' = 5$ gives the correct central charge, $4/5$, for the Potts model. The field content is represented by highest weight states through

$$\alpha = -\alpha_+\lambda_+ - \alpha_-\lambda_- , \tag{3.3}$$

where $\lambda_\pm$ are dominant weights of the algebra parametrized by two pairs of integers, (r_1, r_2) and (s_1, s_2),

$$\lambda_+ = r_1\omega_1 + r_2\omega_2 , \qquad \lambda_- = s_1\omega_1 + s_2\omega_2 , \tag{3.4}$$

and $0 \leq s_{1,2} \leq 3$ and $0 \leq r_{1,2} \leq 2$. For each primary field F there are six αs which all lie in the same Weyl orbit and can be written

$$\begin{aligned}
\alpha_i(F) &= -2\alpha_0 w_i^*\left(-p\Lambda_-(F)\right) , \qquad i = 0, \ldots, 5 , \\
\Lambda_-(F) &= s_1\omega_1 + s_2\omega_2 , \\
\Lambda_+(F) &= r_1\omega_1 + r_2\omega_2 ,
\end{aligned} \tag{3.5}$$

where $r_{1,2}$ and $s_{1,2}$ are read off from the first column of table 1. An equivalent way of generating these is to compute the αs from all three columns of table 1 and form the sets

$$\begin{aligned}
I, \epsilon &: \{\alpha, 4\alpha_0\rho - \alpha\} , \\
\sigma, \psi &: \{\alpha_{\sigma,\psi}, 4\alpha_0\rho - \alpha_{\sigma^\dagger,\psi^\dagger}\} , \\
\sigma^\dagger, \psi^\dagger &: \{\alpha_{\sigma^\dagger,\psi^\dagger}, 4\alpha_0\rho - \alpha_{\sigma,\psi}\} .
\end{aligned} \tag{3.6}$$

The conformal dimensions of the fields obtained this way using (2.19) are the same as for the fields in the three state Potts model which are shown in table 2. This six-fold redundancy in the choice of αs for the primary fields in the bulk theory is a consequence of the property (2.20) and the Z_3 symmetry of the root diagram for A_2 which is in turn responsible for the existence of the chiral W_3 algebra. This changes when we consider boundary conditions which only preserve the Virasoro symmetry and not the full W symmetry. As we will see in section 4, only the Felder complexes (discussed below) built on the first column of table 1 yield W violating boundary states which do not decouple from the bulk theory.

The physical Hilbert space is determined by the BRST structure first elucidated by Felder [11] and extended to the W_3 case in [14, 15]. In this construction it is more convenient to write $\alpha(F)$ in the form

$$\alpha(F) = \alpha_-((1-m_1)\omega_1 + (1-m_2)\omega_2) + \alpha_+((1-n_1)\omega_1 + (1-n_2)\omega_2) \tag{3.7}$$

and to denote the corresponding Fock space by $\mathcal{F}_\alpha = \mathcal{F}^{\mathbf{m}}_{n_1,n_2}$ where $\mathbf{m} = (m_1, m_2)$. The first two BRST charges are

$$\begin{aligned}
Q^{(1)}_k &= e^{i(k-1)\theta}\frac{\sin k\theta}{\sin\theta}\prod_{j=1\ldots k}\oint dz_j V_{\alpha+e_1}(z_j) , \\
Q^{(2)}_k &= e^{i(k-1)\theta}\frac{\sin k\theta}{\sin\theta}\prod_{j=1\ldots k}\oint dz_j V_{\alpha+e_2}(z_j) ,
\end{aligned} \tag{3.8}$$

(s_1, s_2) \ (r_1, r_2)	$(0,0)$	$(0,1)$	$(1,0)$
$(0,0)$	I	$\psi^\dagger$	ψ
$(0,1)$	$\sigma^\dagger$	σ	ϵ
$(1,0)$	σ	ϵ	$\sigma^\dagger$
$(0,2)$	ψ	I	$\psi^\dagger$
$(1,1)$	ϵ	$\sigma^\dagger$	σ
$(2,0)$	$\psi^\dagger$	ψ	I

Table 1: The representations for the primary fields in the 3-state Potts model.

I	σ	$\sigma^\dagger$	ψ	$\psi^\dagger$	ϵ
0	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{5}$

Table 2: The conformal dimensions of the primary fields in the 3-state Potts model.

with the usual Felder contour prescription and where $\theta = 2\pi\alpha_+^2$. These do not (anti-) commute and it is necessary [14] to introduce a third charge $Q_k^{(3)}$ having the property

$$Q_{k_1}^{(3)} Q_{k_2}^{(1)} = Q_{k_1+k_2}^{(1)} Q_{k_1}^{(2)} . \tag{3.9}$$

All the $Q^{(i)}$ commute with the Virasoro algebra by construction and their action on the Fock spaces is

$$\begin{aligned} Q_k^{(1)} \mathcal{F}^{\mathbf{m}}_{n_1,n_2} &= \mathcal{F}^{\mathbf{m}}_{n_1-2k,n_2+k} , \\ Q_k^{(2)} \mathcal{F}^{\mathbf{m}}_{n_1,n_2} &= \mathcal{F}^{\mathbf{m}}_{n_1+k,n_2-2k} , \\ Q_k^{(3)} \mathcal{F}^{\mathbf{m}}_{n_1,n_2} &= \mathcal{F}^{\mathbf{m}}_{n_1-k,n_2-k} . \end{aligned} \tag{3.10}$$

The Felder complex is built up out of the base cell

$$\mathcal{B}_\alpha = \mathcal{B}^{\mathbf{m}}_{n_1,n_2} = \begin{tikzcd}
 & & & \mathcal{F}^{\mathbf{m}}_{n_1,n_2} \arrow[dll, dashed, "Q^{(2)}_{n_2}"'] \arrow[drr, dotted, "Q^{(1)}_{n_1}"] \arrow[ddd, Rightarrow, "Q^{(3)}_{n_1+n_2}"] & & & \\
 & \mathcal{F}^{\mathbf{m}}_{n_1+n_2,-n_2} \arrow[dl, dashed, "Q^{(2)}_{p-n_2}"'] \arrow[d, "Q^{(3)}_{n_1}"] \arrow[drrrr, dotted, "Q^{(1)}_{n_1+n_2}"] & & & & \mathcal{F}^{\mathbf{m}}_{-n_1,n_1+n_2} \arrow[dllll, dashed, "Q^{(2)}_{n_1+n_2}"'] \arrow[d, "Q^{(3)}_{n_2}"'] \arrow[dr, dotted, "Q^{(1)}_{p-n_1}"] & \\
 {} & \mathcal{F}^{\mathbf{m}}_{n_2,-(n_1+n_2)} \arrow[dl, dashed, "Q^{(2)}_{p-n_1-n_2}"'] \arrow[d, Rightarrow, "Q^{(3)}_{p-n_1}"] \arrow[drr, dotted, "Q^{(1)}_{n_2}"] & & & & \mathcal{F}^{\mathbf{m}}_{-(n_1+n_2),n_1} \arrow[dll, dashed, "Q^{(2)}_{n_1}"'] \arrow[d, Rightarrow, "Q^{(3)}_{p-n_2}"] \arrow[dr, dotted, "Q^{(1)}_{p-n_1-n_2}"] & {} \\
 {} & {} & & \mathcal{F}^{\mathbf{m}}_{-n_2,-n_1} \arrow[dl, dashed, "Q^{(2)}_{p-n_1}"'] \arrow[d, "Q^{(3)}_{p-n_1-n_2}"] \arrow[dr, dotted, "Q^{(1)}_{p-n_2}"] & & {} & {} \\
 & & {} & {} & {} & &
\end{tikzcd} \tag{3.11}$$

where maps incoming to the cell have been suppressed. The operation of the $Q^{(i)}$ on the Fock spaces within the base cell corresponds to the operation of the Weyl group on the vectors $n_1\omega_1 + n_2\omega_2$;

$$Q^{(i)} \longleftrightarrow w_i , \tag{3.12}$$

and the operation outside the base cell is the same plus a translation by pe_i. Note that the action of $Q^{(3)}$ shown by the double arrow changes the ghost number by 3 and is excluded

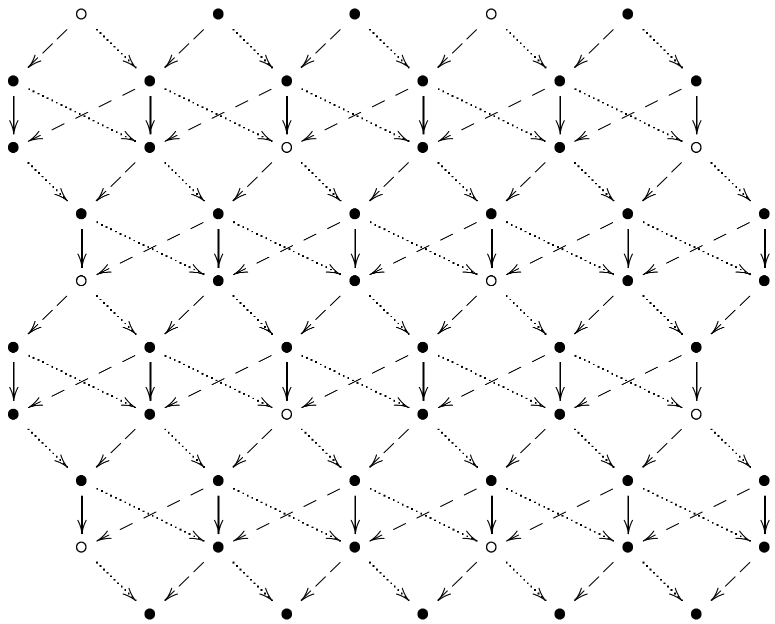

Figure 1: A part of the complex $\mathcal{C}_\alpha$. The open circles correspond to the root of the base cell $\mathcal{F}^{\mathbf{m}}_{n_1,n_2}$ and its images under translations by pe_i. Different types of arrow have the same meaning as in (3.11) and only those which are part of Q_B are shown.

from the BRST charge defined below. It is straightforward to convince oneself that repeated application of the $Q^{(i)}$ generates the infinite complex

$$\mathcal{C}_\alpha = \bigoplus_{M,N\in\mathbb{Z}} \mathcal{B}_{\alpha+p\alpha_+(Me_1+Ne_2)}\,, \tag{3.13}$$

a portion of which is shown in figure 1. It was shown in [14, 15] that a nillpotent BRST operator Q_B graded in the ρ direction can be defined on $\mathcal{C}_\alpha$. To do this we first define operators $d_n^{(i)}$ whose action on the Fock spaces is given by $Q_n^{(i)}$ for $i = 1, 2$, and which increase the ghost number g by one. To keep track of the phases occuring we then define a cocycle operator $\mathcal{N}$ with the properties

$$\mathcal{N}d_n^{(2)} = d_n^{(1)}\mathcal{N} = 0\,, \qquad \mathcal{N}d_n^{(1)} = d_n^{(1)}\,, \qquad d_n^{(2)} = d_n^{(2)}\mathcal{N}\,, \tag{3.14}$$

and

$$d_n^{(3)} = Q_n^{(3)}(-1)^{1+n\mathcal{N}}\,, \tag{3.15}$$

where $[\mathcal{N}, d^{(3)}] = 0$. The BRST operator Q_B for ghost number g is then given by

$$Q_B = d^{(1)} + d^{(2)} + d^{(3)}\frac{1}{2}(1-(-1)^g)\,, \tag{3.16}$$

and has the property $(Q_B)^2 = 0$ when acting on $\mathcal{C}_\alpha$. The Hilbert space $\mathcal{H}$ is then expected to be [11, 14]

$$\mathcal{H} = \frac{\mathrm{Kernel}\,Q_B}{\mathrm{Image}\,Q_B}\,. \tag{3.17}$$

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Therefore W characters for the representations in table 1 can be calculated by computing the expectation value of $\mathrm{Tr}\, q^{L_0-c/24}$ from the alternating sum over the BRST complex [11] with the result

$$\chi_F(q) = \frac{1}{\eta(\tau)^2} \sum_{\substack{w\in\mathcal{W}\\ M,N\in\mathbb{Z}}} \varepsilon_w q^{|p'w(\Lambda_+(F)+\rho)+pp'(Ne_1+Me_2)-p(\Lambda_-(F)+\rho)|^2/2pp'}\,, \tag{3.18}$$

where the Dedekind η function is given by

$$\eta(\tau) = q^{1/24}\prod_{k=1}^{\infty}(1-q^k)\,, \qquad q = e^{i2\pi\tau}\,. \tag{3.19}$$

This agrees with the W_3 characters calculated in other ways (see [13] for a review). For each one of the representations, this formula gives the same as for the corresponding operators in the three state Potts model (see eg [16]) which allows us to conclude that $W_3(4,5)$ does indeed describe the Potts model.

4. Boundary states in free-field representation

Coherent boundary states may be defined in a straightforward generalization of the procedure for the one component Coulomb gas. First we introduce the states $|\alpha,\bar\alpha;\alpha_0\rangle$ which are constructed by applying the vertex operator $V_\alpha(z)$ and its antiholomorphic counterpart, $\overline{V}_{\bar\alpha}(\overline{z})$ to the SL$(2,\mathbb{C})$-invariant vacuum $|0,0;\alpha_0\rangle$,

$$|\alpha,\bar\alpha;\alpha_0\rangle = \lim_{z,\overline{z}\to 0}\overline{V}_{\bar\alpha}(\overline{z})V_\alpha(z)|0,0;\alpha_0\rangle = e^{i\overline{\alpha}\cdot\overline{\phi}_0}e^{i\alpha\cdot\phi_0}|0,0;\alpha_0\rangle\,. \tag{4.1}$$

These states satisfy

$$\begin{aligned} a_0^i|\alpha,\bar\alpha;\alpha_0\rangle &= \alpha^i|\alpha,\bar\alpha;\alpha_0\rangle\,,\\ \bar a_0^i|\alpha,\bar\alpha;\alpha_0\rangle &= \bar\alpha^i|\alpha,\bar\alpha;\alpha_0\rangle\,. \end{aligned} \tag{4.2}$$

The corresponding bra states are given by

$$\langle\alpha,\bar\alpha;\alpha_0| = \langle 0,0;\alpha_0|e^{-i\overline{\alpha}\cdot\overline{\phi}_0}e^{-i\alpha\cdot\phi_0}\,. \tag{4.3}$$

We then make the coherent state ansatz

$$|B(\alpha,\bar\alpha)\rangle = C\,|\alpha,\bar\alpha;\alpha_0\rangle\,, \tag{4.4}$$

$$C = \prod_{k>0}\exp\left(\frac{1}{k}\,a_{-k}\cdot\Lambda\cdot\bar a_{-k}\right), \tag{4.5}$$

where Λ is a matrix to be determined by imposing the boundary condition

$$(L_n - \bar L_{-n})|B(\alpha,\bar\alpha)\rangle = 0\,. \tag{4.6}$$

For positive n we find the constraint

$$\left(\frac{1}{2}\sum_{l=1}^{n-1}\overline{a}_{n-l}\cdot(\Lambda^T\cdot\Lambda-I)\cdot\overline{a}_{-l}+(a_0-2\alpha_0(n+1)\rho)\cdot\Lambda\cdot\overline{a}_{-n}+\right.$$
$$\left.+(-\overline{a}_0-2\alpha_0(n-1)\rho)\cdot\overline{a}_{-n}\right)|\alpha,\bar{\alpha};\alpha_0\rangle\,=\,0\,,\qquad(4.7)$$

where I is the identity matrix, and a similar constraint for negative n. From these we find the conditions

$$\begin{aligned}\Lambda^T\cdot\Lambda&=I\,,\\ \Lambda\cdot\rho+\rho&=0\,,\\ \Lambda^T\cdot\alpha+4\alpha_0\rho-\bar{\alpha}&=0\,.\end{aligned}\qquad(4.8)$$

There are two solutions to (4.8),

$$C=C_{-I}:\quad\Lambda=-I\,,\qquad\overline{\alpha}=4\alpha_0\rho-\alpha\,,\qquad(4.9)$$
$$C=C_{w_\rho}:\quad\Lambda=w_\rho\,,\qquad\overline{\alpha}=4\alpha_0\rho+w_\rho\alpha\,.\qquad(4.10)$$

These allow us to introduce the abbreviated notation

$$|B(\alpha,\Lambda)\rangle=C_\Lambda\,|\alpha,4\alpha_0\rho+\Lambda\cdot\alpha;\alpha_0\rangle\,.\qquad(4.11)$$

The corresponding out-state is

$$\langle B(\alpha,\Lambda)|=\langle\alpha,4\alpha_0\rho+\Lambda\cdot\alpha;\alpha_0|\,C_\Lambda^T\,,\qquad(4.12)$$

where

$$C_\Lambda^T=\prod_{k>0}\exp\left(\frac{1}{k}a_k\cdot\Lambda\cdot\bar{a}_k\right).\qquad(4.13)$$

The conservation of the W current across the boundary can be checked by an explicit calculation. Since $W(z)$ is a primary field of conformal weight $h=3$, we have

$$[L_n,W_m]=(2n-m)W_{m+n}\,.\qquad(4.14)$$

Putting $m=0$ gives

$$W_n=\frac{1}{2n}[L_n,W_0]\,,\qquad(4.15)$$

and so

$$\begin{aligned}(W_n+\bar{W}_{-n})|B(\alpha,\Lambda)\rangle\;&=\;\frac{1}{2n}\left[(L_n-\bar{L}_{-n})(W_0+\bar{W}_0)-(W_0+\bar{W}_0)(L_n-\bar{L}_{-n})\right]|B(\alpha,\Lambda)\rangle\\ &=\;\frac{1}{2n}(L_n-\bar{L}_{-n})(W_0+\bar{W}_0)|B(\alpha,\Lambda)\rangle\,,\end{aligned}\qquad(4.16)$$

where we have used (4.6). After some calculation using (2.24) we find that

$$(W_0+\bar{W}_0)|B(\alpha,-I)\rangle = 0\,, \tag{4.17}$$

but that

$$(W_0+\bar{W}_0)|B(\alpha,w_\rho)\rangle \neq 0\,. \tag{4.18}$$

Thus the W current is conserved for coherent boundary states made with the operator C_{-I} but not for those made with the operator C_{w_ρ}.

Using the free-field representation of the boundary states and the Virasoro operators, we find the cylinder amplitudes

$$\begin{aligned}
\langle B(\alpha,-I)|q^{\frac{1}{2}(L_0+\overline{L}_0-\frac{c}{12})}|B(\beta,-I)\rangle &= \frac{q^{\frac{1}{2}(\beta-2\alpha_0\rho)^2}}{\eta(\tau)^2}\delta_{\alpha,\beta}\,,\\
\langle B(\alpha,w_\rho)|q^{\frac{1}{2}(L_0+\overline{L}_0-\frac{c}{12})}|B(\beta,w_\rho)\rangle &= \frac{q^{\frac{1}{2}(\beta-2\alpha_0\rho)^2}}{\eta(\tau)^2}\delta_{\alpha,\beta}\,,\\
\langle B(\alpha,-I)|q^{\frac{1}{2}(L_0+\overline{L}_0-\frac{c}{12})}|B(\beta,w_\rho)\rangle &= \frac{q^{\frac{1}{2}(\beta-2\alpha_0\rho)^2}}{\eta(2\tau)}\delta_{\alpha,\beta}\delta_{\alpha,-w_\rho\beta}\,.
\end{aligned} \tag{4.19}$$

We see immediately that only those representations for which α is proportional to the Weyl vector ρ have non-zero mixed amplitudes between W conserving and W violating boundary states. It is straightforward to check by examining table 1 that in the Potts model case only I and ϵ have this property and only for the charges α and $4\alpha_0\rho-\alpha$ where the αs are drawn from the first column of table 1. Thus we can construct W conserving boundary states built on the six operators in the three state Potts model and two extra W violating boundary states corresponding to I and ϵ. Linear combinations of these represent the six physical boundary states first found by Cardy [7] plus the free and new boundary states which were only found much later [8]. It was argued in [9] that these eight states are the complete set of boundary states for this model. In the present formalism four other W violating boundary states corresponding to σ, $\sigma^\dagger$, ψ and $\psi^\dagger$ can also be written down but are completely decoupled; thus of course they never appear in the fusion rule analysis of [9].

The states $|B(\alpha,\Lambda)\rangle$ lie in the Fock spaces. To obtain states in the Hilbert space of the conformal field theory we must sum over the Felder complex to obtain

$$\begin{aligned}
|\,F\,\rangle\!\rangle &= \sum_{\substack{w\in\mathcal{W}\\ M,N\in\mathbb{Z}}} \kappa_{wMN}\left|B(-\alpha_+ w^*(\Lambda_+(F))-2\alpha_0 p'p(Ne_1+Me_2)-\alpha_-\Lambda_-(F),-I)\right\rangle\,,\\
|\,\hat{F}\,\rangle\!\rangle &= \sum_{\substack{w\in\mathcal{W}\\ M,N\in\mathbb{Z}}} \kappa_{wMN}\left|B(-\alpha_+ w^*(\Lambda_+(F))-2\alpha_0 p'p(Ne_1+Me_2)-\alpha_-\Lambda_-(F),w_\rho)\right\rangle,
\end{aligned} \tag{4.20}$$

where the constants κ_{wMN} have magnitude 1. There are similar expressions for the bra states but with κ_{wMN} replaced by κ'_{wMN} satisfying

$$\kappa_{wMN}\kappa'_{wMN} = \varepsilon_w\,. \tag{4.21}$$

Using (3.18) and (4.19), we find the cylinder amplitudes

$$\begin{aligned}
\langle\!\langle\, F' \,| q^{\frac{1}{2}(L_0+\overline{L}_0-\frac{c}{12})} |\, F \,\rangle\!\rangle &= \chi_F(q)\delta_{F,F'}\,, \\
\langle\!\langle\, \hat{F}' \,| q^{\frac{1}{2}(L_0+\overline{L}_0-\frac{c}{12})} |\, \hat{F} \,\rangle\!\rangle &= \chi_F(q)\delta_{F,F'}\,, \\
\langle\!\langle\, \hat{I} \,| q^{\frac{1}{2}(L_0+\overline{L}_0-\frac{c}{12})} |\, I \,\rangle\!\rangle &= \frac{q^{-\frac{1}{30}}}{\prod_{k>0}(1-q^{2k})} \sum_{n\in\mathbb{Z}} q^{20n^2+2n} - q^{20n^2+18n+4}\,, \\
\langle\!\langle\, \hat{\epsilon} \,| q^{\frac{1}{2}(L_0+\overline{L}_0-\frac{c}{12})} |\, \epsilon \,\rangle\!\rangle &= \frac{q^{\frac{11}{30}}}{\prod_{k>0}(1-q^{2k})} \sum_{n\in\mathbb{Z}} q^{20n^2+6n} - q^{20n^2+14n+2}\,,
\end{aligned} \tag{4.22}$$

with all other mixed amplitudes being zero.

We can compare these results with the discussion of Affleck et al [8]. The $c = 4/5$ models are Virasoro theories with $(p, q) = (5, 6)$ and on each conformal tower (r, s) can be built *Virasoro* Ishibashi states $|r, s\rangle\!\rangle$ which are related to the characters by

$$\chi_{r,s} = \langle\!\langle r, s| q^{\frac{1}{2}(L_0+\overline{L}_0-\frac{c}{12})} |r, s\rangle\!\rangle\,. \tag{4.23}$$

Following [8] we expect that the cylinder amplitudes for the Potts model should be given in terms of the Virasoro characters by

$$\begin{aligned}
\langle\!\langle\, I \,| q^{\frac{1}{2}(L_0+\overline{L}_0-\frac{c}{12})} |\, I \,\rangle\!\rangle = \langle\!\langle\, \hat{I} \,| q^{\frac{1}{2}(L_0+\overline{L}_0-\frac{c}{12})} |\, \hat{I} \,\rangle\!\rangle &= \chi_{1,1} + \chi_{4,1}\,, \\
\langle\!\langle\, \hat{I} \,| q^{\frac{1}{2}(L_0+\overline{L}_0-\frac{c}{12})} |\, I \,\rangle\!\rangle &= \chi_{1,1} - \chi_{4,1}\,, \\
\langle\!\langle\, \epsilon \,| q^{\frac{1}{2}(L_0+\overline{L}_0-\frac{c}{12})} |\, \epsilon \,\rangle\!\rangle = \langle\!\langle\, \hat{\epsilon} \,| q^{\frac{1}{2}(L_0+\overline{L}_0-\frac{c}{12})} |\, \hat{\epsilon} \,\rangle\!\rangle &= \chi_{2,1} + \chi_{3,1}\,, \\
\langle\!\langle\, \hat{\epsilon} \,| q^{\frac{1}{2}(L_0+\overline{L}_0-\frac{c}{12})} |\, \epsilon \,\rangle\!\rangle &= \chi_{2,1} - \chi_{3,1}\,, \\
\langle\!\langle\, \sigma \,| q^{\frac{1}{2}(L_0+\overline{L}_0-\frac{c}{12})} |\, \sigma \,\rangle\!\rangle = \langle\!\langle\, \sigma^\dagger \,| q^{\frac{1}{2}(L_0+\overline{L}_0-\frac{c}{12})} |\, \sigma^\dagger \,\rangle\!\rangle &= \chi_{3,3}\,, \\
\langle\!\langle\, \psi \,| q^{\frac{1}{2}(L_0+\overline{L}_0-\frac{c}{12})} |\, \psi \,\rangle\!\rangle = \langle\!\langle\, \psi^\dagger \,| q^{\frac{1}{2}(L_0+\overline{L}_0-\frac{c}{12})} |\, \psi^\dagger \,\rangle\!\rangle &= \chi_{4,3}\,.
\end{aligned} \tag{4.24}$$

It can be checked that the identities between Riemann-Jacobi functions implied by these relationships and our results (4.22) are all true; as an example we give a proof in appendix B for a case which has not appeared in the literature before, namely the mixed amplitude for the identity operator.

In terms of these Ishibashi states Cardy's physical boundary states [7] take the usual form for the three state Potts model,

$$\begin{aligned}
|\tilde{I}\rangle &= N\left\{(|I\rangle\!\rangle + |\psi\rangle\!\rangle + |\psi^\dagger\rangle\!\rangle) + \lambda(|\epsilon\rangle\!\rangle + |\sigma\rangle\!\rangle + |\sigma^\dagger\rangle\!\rangle)\right\}, \\
|\tilde{\epsilon}\rangle &= N\left\{\lambda^2(|I\rangle\!\rangle + |\psi\rangle\!\rangle + |\psi^\dagger\rangle\!\rangle) - \lambda^{-1}(|\epsilon\rangle\!\rangle + |\sigma\rangle\!\rangle + |\sigma^\dagger\rangle\!\rangle)\right\}, \\
|\tilde{\psi}\rangle &= N\left\{(|I\rangle\!\rangle + \omega|\psi\rangle\!\rangle + \bar{\omega}|\psi^\dagger\rangle\!\rangle) + \lambda(|\epsilon\rangle\!\rangle + \omega|\sigma\rangle\!\rangle + \bar{\omega}|\sigma^\dagger\rangle\!\rangle)\right\}, \\
|\tilde{\psi}^\dagger\rangle &= N\left\{(|I\rangle\!\rangle + \bar{\omega}|\psi\rangle\!\rangle + \omega|\psi^\dagger\rangle\!\rangle) + \lambda(|\epsilon\rangle\!\rangle + \bar{\omega}|\sigma\rangle\!\rangle + \omega|\sigma^\dagger\rangle\!\rangle)\right\}, \\
|\tilde{\sigma}\rangle &= N\left\{\lambda^2(|I\rangle\!\rangle + \omega|\psi\rangle\!\rangle + \bar{\omega}|\psi^\dagger\rangle\!\rangle) - \lambda^{-1}(|\epsilon\rangle\!\rangle + \omega|\sigma\rangle\!\rangle + \bar{\omega}|\sigma^\dagger\rangle\!\rangle)\right\}, \\
|\tilde{\sigma}^\dagger\rangle &= N\left\{\lambda^2(|I\rangle\!\rangle + \bar{\omega}|\psi\rangle\!\rangle + \omega|\psi^\dagger\rangle\!\rangle) - \lambda^{-1}(|\epsilon\rangle\!\rangle + \bar{\omega}|\sigma\rangle\!\rangle + \omega|\sigma^\dagger\rangle\!\rangle)\right\},
\end{aligned} \tag{4.25}$$

where

$$N = \sqrt{\frac{2}{\sqrt{15}} \sin\frac{\pi}{5}}\,, \qquad \lambda = \sqrt{\frac{\sin(2\pi/5)}{\sin(\pi/5)}}\,, \tag{4.26}$$

and

$$\omega = e^{2\pi i/3}\,, \qquad \bar{\omega} = e^{4\pi i/3}\,. \tag{4.27}$$

These states represent the fixed and mixed boundary conditions for the Potts model spins. In addition there are two more physical boundary states

$$\begin{aligned} |\text{free}\rangle &= N\sqrt{3}\left(|\,\hat{I}\,\rangle\!\rangle - \lambda|\,\hat{\epsilon}\,\rangle\!\rangle\right), \\ |\text{new}\rangle &= N\sqrt{3}\left(\lambda^2|\,\hat{I}\,\rangle\!\rangle + \lambda^{-1}|\,\hat{\epsilon}\,\rangle\!\rangle\right), \end{aligned} \tag{4.28}$$

which represent the free and so-called new boundary conditions for the spin model.

5. Critical Potts model correlation functions with boundary

We will now show how the free-field formalism can be used to compute exact correlation functions of bulk operators in the presence of a boundary when the boundary condition is represented by the states (4.25), (4.28).

5.1 General considerations

We consider a unit disk in the z-plane, where a boundary with condition $\tilde{\alpha}$ is placed at $|z| = 1$. At the origin we place the Möbius invariant vacuum $|0, 0; \alpha_0\rangle$. Then p-point correlation functions on the disk are given by

$$\langle\tilde{\alpha}|V_{\alpha_1}(z_1)\bar{V}_{\bar{\alpha}_1}(\bar{z}_1)\cdots V_{\alpha_p}(z_p)\bar{V}_{\bar{\alpha}_p}(\bar{z}_p) \times (\text{screening operators})|0, 0; \alpha_0\rangle\,. \tag{5.1}$$

As a Cardy state $\langle\tilde{\alpha}|$ is a linear sum of the Ishibashi states $\langle\!\langle F|$ and $\langle\!\langle \hat{F}|$ which are in turn linear sums of the coherent states $\langle B(\alpha, \Lambda)|$, finding the correlation functions (5.1) boils down to evaluating the disk amplitudes with boundary charges α and boundary types Λ,

$$\begin{aligned} &\langle B(\alpha, \Lambda)|V_{\alpha_1}(z_1)\bar{V}_{\bar{\alpha}_1}(\bar{z}_1)\cdots V_{\alpha_p}(z_p)\bar{V}_{\bar{\alpha}_p}(\bar{z}_p) \\ &\times (Q_+^{(1)})^{m_1}(Q_-^{(1)})^{n_1}(Q_+^{(2)})^{m_2}(Q_-^{(2)})^{n_2}(\bar{Q}_+^{(1)})^{\bar{m}_1}(\bar{Q}_-^{(1)})^{\bar{n}_1}(\bar{Q}_+^{(2)})^{\bar{m}_2}(\bar{Q}_-^{(2)})^{\bar{n}_2}|0, 0; \alpha_0\rangle\,, \end{aligned} \tag{5.2}$$

where m_1, etc. are numbers of the screening operators. Physical correlation functions are simply linear combinations of the amplitudes (5.2). Since the inner products of the Fock spaces are non-vanishing only when both the holomorphic and antiholomorphic charge neutrality conditions are satisfied, the amplitudes (5.2) are subject to the conditions,

$$-\alpha + \alpha_1 + \cdots + \alpha_p + m_1\alpha_+ e_1 + m_2\alpha_+ e_2 + n_1\alpha_- e_1 + n_2\alpha_- e_2 = 0\,, \tag{5.3}$$

$$-4\alpha_0\rho - \Lambda\cdot\alpha + \bar{\alpha}_1 + \cdots + \bar{\alpha}_p + \bar{m}_1\alpha_+ e_1 + \bar{m}_2\alpha_+ e_2 + \bar{n}_1\alpha_- e_1 + \bar{n}_2\alpha_- e_2 = 0\,, \tag{5.4}$$

otherwise they vanish. These neutrality conditions relate the boundary charges α to possible configurations of the screening operators. The disk amplitudes are evaluated using the formula,

$$\langle B(\alpha,\Lambda)|\prod_{i=1}^{M}V_{\alpha_i}(z_i)\prod_{j=1}^{N}\bar{V}_{\overline{\alpha}_j}(\bar{z}_j)|0,0;\alpha_0\rangle = \delta_{\alpha,\sum\alpha_i}\delta_{4\alpha_0\rho+\Lambda\cdot\alpha,\sum\overline{\alpha}_j}\times$$
$$\times\left\{\prod_{i<j}^{M}(z_i-z_j)^{\alpha_i\cdot\alpha_j}\right\}\left\{\prod_{i<j}^{N}(\bar{z}_i-\bar{z}_j)^{\overline{\alpha}_i\cdot\overline{\alpha}_j}\right\}\times$$
$$\times\left\{\prod_{i=1}^{M}\prod_{j=1}^{N}(1-z_i\bar{z}_j)^{-\alpha_i\cdot\Lambda\cdot\overline{\alpha}_j}\right\}. \tag{5.5}$$

The presence of screening operators leads to integral representations for the amplitudes. The integration contours are originally Felder's contours (concentric circles around the origin and attached to vertex operators). It is argued [5] that at least in the cases of 1 and 2-point functions these contours can be deformed suitably to give convergent functions which correspond to certain conformal blocks. We will now use this simple setup to compute Potts model correlation functions.

5.2 Vertex operators

In order to compute disk correlation functions in the Coulomb gas formalism, we need to define non-chiral primary operators in terms of chiral operators and then represent them with vertex operators.

The first step involves some intricacies because the non-chiral spin operators for instance can be either

$$\sigma(z,\bar{z})=\sigma(z)\otimes\bar{\sigma}(\bar{z})\,,\qquad \sigma^\dagger(z,\bar{z})=\sigma^\dagger(z)\otimes\bar{\sigma}^\dagger(\bar{z})\,, \tag{5.6}$$

or

$$\sigma(z,\bar{z})=\sigma(z)\otimes\bar{\sigma}^\dagger(\bar{z})\,,\qquad \sigma^\dagger(z,\bar{z})=\sigma^\dagger(z)\otimes\bar{\sigma}(\bar{z})\,. \tag{5.7}$$

Although the first one (5.6) might seem more natural, this does not represent the Potts model correctly since it would lead to vanishing spin 1-point functions even near a fixed boundary. Thus we shall adopt the definition (5.7) for the non-chiral spin operators and similarly we define

$$\psi(z,\bar{z})=\psi(z)\otimes\bar{\psi}^\dagger(\bar{z})\,,\qquad \psi^\dagger(z,\bar{z})=\psi^\dagger(z)\otimes\bar{\psi}(\bar{z})\,. \tag{5.8}$$

The identity and energy operators do not have this ambiguity and we shall simply define

$$I(z,\bar{z})=I(z)\otimes\bar{I}(\bar{z})\,,\qquad \epsilon(z,\bar{z})=\epsilon(z)\otimes\bar{\epsilon}(\bar{z})\,. \tag{5.9}$$

In the second step, we need to represent each W primary field by a vertex operator having one of the six charges of (3.5). In the bulk theory and also in presence of a W conserving boundary, one may allocate whichever of the six charges is most convenient as they should be all equivalent under the chiral algebra. However, as we have seen, in the presence of W violating boundary conditions this equivalence breaks down and only charges associated with the first column of table 1 are allowed. We are still free to use α or its conjugate $4\alpha_0\rho + w_\rho\alpha$ to construct a chiral vertex operator and can, in particular, represent a non-chiral operator as a product of different holomorphic and antiholomorphic vertex operators, e.g. $I(z,\bar{z}) = V_0(z) \otimes \bar{V}_{4\alpha_0\rho}(\bar{z})$. In practice we shall choose a set of vertex operators which minimizes the number of screening operators.

5.3 One point functions

In the case of the spin 1-point function $\langle\sigma(z,\bar{z})\rangle_{\text{boundary}}$ with various boundary conditions represented by the Cardy states, the number of screening operators is minimized by defining $\sigma(z,\bar{z}) = V_{\alpha_1}(z)\otimes\bar{V}_{4\alpha_0\rho-\alpha_1}(\bar{z})$, where α_1 is either $\alpha_1 = -\alpha_-\omega_1$ or its conjugate $4\alpha_0\rho+\alpha_-\omega_2$. Then the 1-point function is a linear sum of the disk 1-point amplitudes,

$$\begin{aligned} &\langle B(\alpha,\Lambda)|V_{\alpha_1}(z_1)\bar{V}_{4\alpha_0\rho-\alpha_1}(\bar{z}_1)(Q_+^{(1)})^{m_1}(Q_-^{(1)})^{n_1} \times \\ &\quad\times(Q_+^{(2)})^{m_2}(Q_-^{(2)})^{n_2}(\bar{Q}_+^{(1)})^{\bar{m}_1}(\bar{Q}_-^{(1)})^{\bar{n}_1}(\bar{Q}_+^{(2)})^{\bar{m}_2}(\bar{Q}_-^{(2)})^{\bar{n}_2}|0,0;\alpha_0\rangle\,, \end{aligned} \tag{5.10}$$

subject to the charge neutrality conditions

$$-\alpha + \alpha_1 + m_1\alpha_+e_1 + m_2\alpha_+e_2 + n_1\alpha_-e_1 + n_2\alpha_-e_2 = 0\,, \tag{5.11}$$

$$-\Lambda\cdot\alpha - \alpha_1 + \bar{m}_1\alpha_+e_1 + \bar{m}_2\alpha_+e_2 + \bar{n}_1\alpha_-e_1 + \bar{n}_2\alpha_-e_2 = 0\,. \tag{5.12}$$

Adding the above two expressions we have

$$(I+\Lambda)\cdot\alpha = (m_1+\bar{m}_1)\alpha_+e_1 + (m_2+\bar{m}_2)\alpha_+e_2 + (n_1+\bar{n}_1)\alpha_-e_1 + (n_2+\bar{n}_2)\alpha_-e_2\,. \tag{5.13}$$

For $\Lambda = -I$, this condition is satisfied by $m_1 = \bar{m}_1 = m_2 = \bar{m}_2 = n_1 = \bar{n}_1 = n_2 = \bar{n}_2 = 0$, that is, with no screening operators, and from (5.11) we have $\alpha = \alpha_1$. Thus in this case,

$$\langle B(\alpha,-I)|V_{\alpha_1}(z)\bar{V}_{4\alpha_0\rho-\alpha_1}(\bar{z})|0,0;\alpha_0\rangle = (1-z\bar{z})^{-2h}\delta_{\alpha,\alpha_1}\,, \tag{5.14}$$

where $h = \frac{1}{2}\alpha_1\cdot(\alpha_1 - 4\alpha_0\rho) = 1/15$, is the only non-trivial disk amplitude. This means that the charge neutrality condition picks up the coefficient of $\langle B(\alpha_1,-I)|$ from the boundary state. When $\Lambda = w_\rho$, the left hand side of (5.13) is proportional to $e_1 - e_2$ and it cannot be neutralized with non-negative powers of the screening operators. The only exception is when $(I + w_\rho)\alpha = 0$, but this does not happen in the case of the spin 1-point function. Thus the charge neutrality can never be satisfied for $\Lambda = w_\rho$ and we have no contribution from $\langle B(\alpha, w_\rho)|$ boundary states.

Thus, for example, the spin 1-point function with the boundary condition $\tilde{I}$ will be (after normalization)

$$\frac{\langle\tilde{I}|V_{\alpha_1}(z)\bar{V}_{4\alpha_0\rho-\alpha_1}(\bar{z})|0,0;\alpha_0\rangle}{\langle\tilde{I}|0,0;\alpha_0\rangle} = \lambda(1-z\bar{z})^{-2/15}\,. \tag{5.15}$$

This 1-point function on the disk is conveniently mapped onto the half-plane, via the global conformal transformation,

$$w = -iy_0\frac{z-1}{z+1}\,, \qquad \bar{w} = iy_0\frac{\bar{z}-1}{\bar{z}+1}\,, \tag{5.16}$$

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which takes the origin $(z,\bar{z})=(0,0)$ to $(w,\bar{w})=(iy_0,-iy_0)$. The spin 1-point function on the half-plane is then,

$$\langle\sigma\rangle_{\tilde{I}} = \lambda(2y)^{-2/15}\,, \tag{5.17}$$

where y is the distance from the boundary. Similarly the spin 1-point function with other boundary conditions is (on the half-plane),

$$\begin{aligned}
\langle\sigma\rangle_{\tilde{\psi}} &= \omega\lambda(2y)^{-2/15}\,, && (5.18)\\
\langle\sigma\rangle_{\tilde{\psi}^\dagger} &= \bar{\omega}\lambda(2y)^{-2/15}\,, && (5.19)\\
\langle\sigma\rangle_{\tilde{\epsilon}} &= -\frac{1}{\lambda^3}(2y)^{-2/15}\,, && (5.20)\\
\langle\sigma\rangle_{\tilde{\sigma}} &= -\frac{\omega}{\lambda^3}(2y)^{-2/15}\,, && (5.21)\\
\langle\sigma\rangle_{\tilde{\sigma}^\dagger} &= -\frac{\bar{\omega}}{\lambda^3}(2y)^{-2/15}\,, && (5.22)\\
\langle\sigma\rangle_{\text{free}} &= 0\,, && (5.23)\\
\langle\sigma\rangle_{\text{new}} &= 0. && (5.24)
\end{aligned}$$

The first six Cardy states are identified as the fixed and mixed boundary conditions,

$$\begin{aligned}
\text{Fixed:}\quad & \tilde{I}=(A)\,, \qquad \tilde{\psi}=(B)\,, \qquad \tilde{\psi}^\dagger=(C)\,, && (5.25)\\
\text{Mixed:}\quad & \tilde{\epsilon}=(BC)\,, \qquad \tilde{\sigma}=(CA)\,, \qquad \tilde{\sigma}^\dagger=(AB)\,. && (5.26)
\end{aligned}$$

The free and new boundary conditions are not immediately distinguishable from the spin 1-point function only.

Next we consider the energy 1-point function. For definiteness we let $\epsilon(z,\bar{z}) = V_{\alpha_1}(z)\otimes\bar{V}_{4\alpha_0\rho-\alpha_1}(\bar{z})$, with $\alpha_1=-\alpha_-(\omega_1+\omega_2)$. The computation is almost the same as above. For $\Lambda=-I$ boundary, the charge neutrality condition is satisfied with no screening operators, and (5.14) with $h=2/5$ is the only non-trivial contribution in this case. However, the charge neutrality condition is fulfilled for $\Lambda=w_\rho$ as well, with $\alpha=\alpha_1$ and no screening operators. Thus the amplitude

$$\langle B(\alpha,w_\rho)|V_{\alpha_1}(z)\bar{V}_{4\alpha_0\rho-\alpha_1}(\bar{z})|0,0;\alpha_0\rangle = (1-z\bar{z})^{-4/5}\delta_{\alpha,\alpha_1} \tag{5.27}$$

also contributes in this case. The resulting energy 1-point function for various boundary conditions are (on the half plane)

$$\begin{aligned}
\langle\epsilon\rangle_{\tilde{I}} = \langle\epsilon\rangle_{\tilde{\psi}} = \langle\epsilon\rangle_{\tilde{\psi}^\dagger} &= \lambda(2y)^{-4/5}\,, && (5.28)\\
\langle\epsilon\rangle_{\tilde{\epsilon}} = \langle\epsilon\rangle_{\tilde{\sigma}} = \langle\epsilon\rangle_{\tilde{\sigma}^\dagger} &= -\frac{1}{\lambda^3}(2y)^{-4/5}\,, && (5.29)\\
\langle\epsilon\rangle_{\text{free}} &= -\lambda(2y)^{-4/5}\,, && (5.30)\\
\langle\epsilon\rangle_{\text{new}} &= \frac{1}{\lambda^3}(2y)^{-4/5}\,. && (5.31)
\end{aligned}$$

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As we have seen, the 1-point functions come along with the coefficients of the corresponding Ishibashi states in the Cardy states, and this is in agreement with the commonly accepted understanding[1] that such coefficients are essentially the 1-point bulk-brane coupling constants [17].

5.4 Spin two point functions

Next, we consider the spin 2-point functions

$$\langle \sigma(z_1,\bar{z}_1)\sigma(z_2,\bar{z}_2)\rangle_{\text{boundary}} \tag{5.32}$$

and

$$\langle \sigma^\dagger(z_1,\bar{z}_1)\sigma(z_2,\bar{z}_2)\rangle_{\text{boundary}} \tag{5.33}$$

and find their exact forms for various boundary conditions. In principle they should also be obtainable by the conventional approach [3]; that is, by solving a 6th order differential equation arising from the singular vector at level 6,

$$\begin{aligned}\Big(L_{-6} - \frac{96367}{136856}L_{-1}L_{-5} - \frac{1630}{17107}L_{-2}L_{-4} + \frac{33795}{136856}L^2_{-1}L_{-4} - \frac{2437}{102642}L^2_{-3} + \\ + \frac{4909}{68428}L_{-1}L_{-2}L_{-3} - \frac{16905}{273712}L^3_{-1}L_{-3} + \frac{576}{85535}L^3_{-2} - \frac{651}{34214}L^2_{-1}L^2_{-2} + \\ + \frac{3465}{273712}L^4_{-1}L_{-2} - \frac{675}{547424}L^6_{-1}\Big)\,|\phi_{2,3}\rangle\,,\end{aligned} \tag{5.34}$$

and then fixing their coefficients using the solutions of sewing relations. The Coulomb gas calculation presented below is equivalent, but practically it is considerably simpler.

Let us consider $\langle\sigma\sigma\rangle_{\text{boundary}}$ first. We represent the primary operators, $\sigma(z_1,\bar{z}_1) = V_1(z_1)\otimes\bar{V}_1(\bar{z}_1)$, $\sigma(z_2,\bar{z}_2) = V_2(z_2)\otimes\bar{V}_2(\bar{z}_2)$, by the vertex operators having charges,

$$\alpha_1 = -\alpha_-\omega_1\,, \tag{5.35}$$

$$\alpha_2 = -\alpha_-\omega_1\,, \tag{5.36}$$

$$\bar{\alpha}_1 = -\alpha_-\omega_2\,, \tag{5.37}$$

$$\bar{\alpha}_2 = 4\alpha_0\rho + \alpha_-\omega_1. \tag{5.38}$$

The holomorphic and antiholomorphic charge neutrality conditions on the disk are,

$$-\alpha - 2\alpha_-\omega_1 + m_1\alpha_+e_1 + m_2\alpha_+e_2 + n_1\alpha_-e_1 + n_2\alpha_-e_2 = 0\,, \tag{5.39}$$

$$-\Lambda\cdot\alpha + \alpha_-(\omega_1-\omega_2) + \bar{m}_1\alpha_+e_1 + \bar{m}_2\alpha_+e_2 + \bar{n}_1\alpha_-e_1 + \bar{n}_2\alpha_-e_2 = 0\,. \tag{5.40}$$

Summing (5.39) and (5.40), we have

$$\begin{aligned}(m_1+\bar{m}_1)\alpha_+e_1 + (m_2+\bar{m}_2)\alpha_+e_2 + (n_1+\bar{n}_1)\alpha_-e_1 + (n_2+\bar{n}_2)\alpha_-e_2 = \\ = (I+\Lambda)\cdot\alpha + \alpha_-(\omega_1+\omega_2) \\ = (I+\Lambda)\cdot\alpha + \alpha_-(e_1+e_2)\,.\end{aligned} \tag{5.41}$$

[1] In our notation, which is traditional in the literature, the bra-Ishibashi state $\langle\!\langle\sigma|$ couples to σ but it is $\sigma^\dagger$ which couples to the ket-Ishibashi state $|\sigma\rangle\!\rangle$. In this sense it would be more natural to exchange the definitions of $|\sigma\rangle\!\rangle$ and $|\sigma^\dagger\rangle\!\rangle$.

It can be shown that when $\Lambda = w_\rho$ the charge neutrality is not satisfied. For $\Lambda = -I$, (5.41) implies

$$m_1 = \bar{m}_1 = m_2 = \bar{m}_2 = 0\,, \qquad n_1 + \bar{n}_1 = 1\,, \qquad n_2 + \bar{n}_2 = 1\,. \tag{5.42}$$

There are potentially four cases where the charge neutrality is satisfied,

$$\begin{array}{rlll}
(\mathrm{I}): & n_1 = n_2 = 1\,, & \bar{n}_1 = \bar{n}_2 = 0\,, & \alpha = \alpha_-(\omega_2 - \omega_1)\,, \\
(\mathrm{II}): & n_1 = \bar{n}_2 = 1\,, & \bar{n}_1 = n_2 = 0\,, & \alpha = -\alpha_-\omega_2\,, \\
(\mathrm{III}): & \bar{n}_1 = n_2 = 1\,, & n_1 = \bar{n}_2 = 0\,, & \alpha = \alpha_-(2\omega_2 - 3\omega_1)\,, \\
(\mathrm{IV}): & \bar{n}_1 = \bar{n}_2 = 1\,, & n_1 = n_2 = 0\,, & \alpha = -2\alpha_-\omega_1\,.
\end{array} \tag{5.43}$$

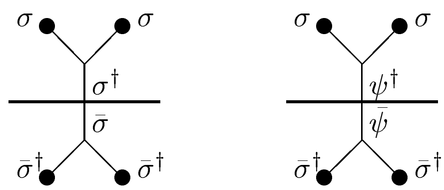
We notice that the charge configurations (I) and (III) can never be realized since the corresponding boundary charges do not exist in the Cardy states. The boundary charges of (II) and (IV) are included in the Ishibashi states $\langle\langle\sigma^\dagger|$ and $\langle\langle\psi^\dagger|$ respectively, indicating which intermediate state is reflected by the boundary. This of course is in agreement with the fusion rules,

$$\sigma \times \sigma = \psi^\dagger + \sigma^\dagger\,, \tag{5.44}$$

$$\sigma^\dagger \times \sigma^\dagger = \psi + \sigma\,. \tag{5.45}$$

We shall denote the disc 2-point amplitudes corresponding to (II) and (IV) as I_σ and I_ψ, respectively. They correspond to the two conformal blocks depicted in the figure below.

Conformal block I_σ Conformal block I_ψ



The evaluation of the amplitudes I_σ and I_ψ is straightforward. As (II) involves $Q_-^{(1)}$ in the holomorphic and $\bar{Q}_-^{(2)}$ in the antiholomorphic sector, we have

$$\begin{aligned}
I_\sigma &= \langle B(-\alpha_-\omega_2, -I)|V_1(z_1)\bar{V}_1(\bar{z}_1)Q_-^{(1)}\bar{Q}_-^{(2)}V_2(0)\bar{V}_2(0)|0,0;\alpha_0\rangle \qquad (5.46)\\
&= z_1^{8/15}\bar{z}_1^{-2/15}(1 - z_1\bar{z}_1)^{4/15}\oint du \oint d\bar{v}(z_1 - u)^{-4/5}u^{-4/5}(\bar{z}_1 - \bar{v})^{-4/5}\bar{v}^{-2/5}(1 - u\bar{v})^{-4/5}\,.
\end{aligned}$$

We have used the global conformal invariance to set $z_2 = \bar{z}_2 = 0$ (this is equivalent to sending one of the four points to zero and another to infinity in the corresponding chiral four point function). As the screening operator $Q_-^{(1)}$ lies in the holomorphic sector, the convergent integral must be proportional to the u-integral between z_1 and z_2. Similarly, the antiholomorphic screening operator $\bar{Q}_-^{(2)}$ leads to $\bar{v}$-integration between $\bar{z}_1$ and $\bar{z}_2$. Hence the integration contours may be deformed into

$$\oint du \oint d\bar{v} \to \int_{z_2}^{z_1} du \int_{\bar{z}_2}^{\bar{z}_1} d\bar{v}\,, \tag{5.47}$$

and we have

$$I_\sigma = N_\sigma \xi^{-1/15}(1-\xi)^{4/15} F\left(\frac{1}{5},\frac{3}{5},\frac{2}{5};\xi\right), \tag{5.48}$$

where $\xi = z_1\bar{z}_1$. The normalization of the conformal block is fixed by its off-boundary limit and the constant N_σ is identified with the 3-point coupling constant,

$$N_\sigma = C_{\sigma\sigma}{}^{\sigma^\dagger} = \frac{\Gamma(1/5)\Gamma(3/5)}{\Gamma(2/5)^2\lambda}, \tag{5.49}$$

appearing in the bulk OPE [18],

$$\sigma(z_1,\bar{z}_1)\sigma(z_2,\bar{z}_2) = C_{\sigma\sigma}{}^{\sigma^\dagger}|z_1-z_2|^{-2/15}\sigma^\dagger(z_2,\bar{z}_2) + C_{\sigma\sigma}{}^{\psi^\dagger}|z_1-z_2|^{16/15}\psi^\dagger(z_2,\bar{z}_2)+\cdots. \tag{5.50}$$

The other conformal block I_ψ is similarly evaluated as

$$\begin{aligned} I_\psi &= \langle B(-2\alpha_-\omega_1,-I)|V_1(z_1)\bar{V}_1(\bar{z}_1)\bar{Q}_-^{(1)}\bar{Q}_-^{(2)}V_2(0)\bar{V}_2(0)|0,0;\alpha_0\rangle \\ &= N_\psi \xi^{8/15}(1-\xi)^{4/15} F\left(\frac{4}{5},\frac{6}{5},\frac{8}{5};\xi\right), \end{aligned} \tag{5.51}$$

where

$$N_\psi = C_{\sigma\sigma}{}^{\psi^\dagger} = \frac{1}{3}. \tag{5.52}$$

The spin 2-point function $\langle\sigma\sigma\rangle_{\mathrm{boundary}}$ with various boundary conditions is then given by linear combinations of I_σ and I_ψ, with the coefficients of the Ishibashi states $\langle\langle\sigma^\dagger|$ and $\langle\langle\psi^\dagger|$ picked out from the Cardy states. On the disk we find

$$(A): \quad \frac{\langle\tilde{I}|\sigma(z_1,\bar{z}_1)\sigma(0,0)|0,0;\alpha_0\rangle}{\langle\tilde{I}|0,0;\alpha_0\rangle} = I_\psi + \lambda I_\sigma, \tag{5.53}$$

$$(B): \quad \frac{\langle\tilde{\psi}|\sigma(z_1,\bar{z}_1)\sigma(0,0)|0,0;\alpha_0\rangle}{\langle\tilde{\psi}|0,0;\alpha_0\rangle} = \bar{\omega}(I_\psi + \lambda I_\sigma), \tag{5.54}$$

$$(C): \quad \frac{\langle\tilde{\psi}^\dagger|\sigma(z_1,\bar{z}_1)\sigma(0,0)|0,0;\alpha_0\rangle}{\langle\tilde{\psi}^\dagger|0,0;\alpha_0\rangle} = \omega(I_\psi + \lambda I_\sigma), \tag{5.55}$$

$$(BC): \quad \frac{\langle\tilde{\epsilon}|\sigma(z_1,\bar{z}_1)\sigma(0,0)|0,0;\alpha_0\rangle}{\langle\tilde{\epsilon}|0,0;\alpha_0\rangle} = I_\psi - \frac{1}{\lambda^3}I_\sigma, \tag{5.56}$$

$$(CA): \quad \frac{\langle\tilde{\sigma}|\sigma(z_1,\bar{z}_1)\sigma(0,0)|0,0;\alpha_0\rangle}{\langle\tilde{\sigma}|0,0;\alpha_0\rangle} = \bar{\omega}(I_\psi - \frac{1}{\lambda^3}I_\sigma), \tag{5.57}$$

$$(AB): \quad \frac{\langle\tilde{\sigma}^\dagger|\sigma(z_1,\bar{z}_1)\sigma(0,0)|0,0;\alpha_0\rangle}{\langle\tilde{\sigma}^\dagger|0,0,\alpha_0\rangle} = \omega(I_\psi - \frac{1}{\lambda^3}I_\sigma), \tag{5.58}$$

$$(\mathrm{free}): \quad \frac{\langle\mathrm{free}|\sigma(z_1,\bar{z}_1)\sigma(0,0)|0,0;\alpha_0\rangle}{\langle\mathrm{free}|0,0;\alpha_0\rangle} = 0, \tag{5.59}$$

$$(\mathrm{new}): \quad \frac{\langle\mathrm{new}|\sigma(z_1,\bar{z}_1)\sigma(0,0)|0,0;\alpha_0\rangle}{\langle\mathrm{new}|0,0;\alpha_0\rangle} = 0, \tag{5.60}$$

with I_σ and I_ψ given by (5.48) and (5.51).

The other type of 2-point function $\langle\sigma^\dagger\sigma\rangle_{\mathrm{boundary}}$ can be computed in a similar manner, but this time the W violating boundary $\langle B(\alpha,w_\rho)|$ gives non-trivial amplitudes. There are

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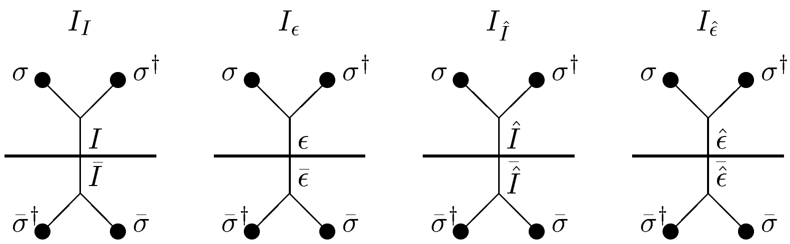
four conformal blocks:



which are found to be

$$I_I = N_I \xi^{-2/15}(1-\xi)^{4/15} F\left(\frac{4}{5}, \frac{1}{5}, \frac{3}{5}; \xi\right), \tag{5.61}$$

$$I_\epsilon = N_\epsilon \xi^{4/15}(1-\xi)^{4/15} F\left(\frac{6}{5}, \frac{3}{5}, \frac{7}{5}; \xi\right), \tag{5.62}$$

$$I_{\hat{I}} = N_{\hat{I}} \xi^{-2/15}(1-\xi)^{8/15} F\left(\frac{4}{5}, \frac{2}{5}, \frac{3}{5}; \xi\right), \tag{5.63}$$

$$I_{\hat{\epsilon}} = N_{\hat{\epsilon}} \xi^{4/15}(1-\xi)^{8/15} F\left(\frac{4}{5}, \frac{6}{5}, \frac{7}{5}; \xi\right). \tag{5.64}$$

The normalization constants are determined as

$$N_I = N_{\hat{I}} = 1\,, \tag{5.65}$$

$$N_\epsilon = N_{\hat{\epsilon}} = C_{\sigma^\dagger\sigma}{}^{\epsilon} = \frac{\Gamma(1/5)\Gamma(3/5)}{2\Gamma(2/5)^2\lambda}, \tag{5.66}$$

where $C_{\sigma^\dagger\sigma}{}^{\epsilon}$ is the 3-point coupling constant in the OPE,

$$\sigma^\dagger(z_1,\bar{z}_1)\sigma(z_2,\bar{z}_2) = |z_1-z_2|^{-4/15} + C_{\sigma^\dagger\sigma}{}^{\epsilon}|z_1-z_2|^{8/15}\epsilon(w,\bar{w}) + \cdots\,. \tag{5.67}$$

The 2-point function $\langle\sigma^\dagger\sigma\rangle_{\text{boundary}}$ for the eight boundary conditions is,

$$(\text{fixed}):\ \frac{\langle\tilde{I},\tilde{\psi},\tilde{\psi}^\dagger|\sigma^\dagger(z_1,\bar{z}_1)\sigma(0,0)|0,0;\alpha_0\rangle}{\langle\tilde{I},\tilde{\psi},\tilde{\psi}^\dagger|0,0;\alpha_0\rangle} = I_I + \lambda I_\epsilon\,, \tag{5.68}$$

$$(\text{mixed}):\ \frac{\langle\tilde{\epsilon},\tilde{\sigma},\tilde{\sigma}^\dagger|\sigma^\dagger(z_1,\bar{z}_1)\sigma(0,0)|0,0;\alpha_0\rangle}{\langle\tilde{\epsilon},\tilde{\sigma},\tilde{\sigma}^\dagger|0,0;\alpha_0\rangle} = I_I - \frac{1}{\lambda^3}I_\epsilon\,, \tag{5.69}$$

$$(\text{free}):\ \frac{\langle\text{free}|\sigma^\dagger(z_1,\bar{z}_1)\sigma(0,0)|0,0;\alpha_0\rangle}{\langle\text{free}|0,0;\alpha_0\rangle} = I_{\hat{I}} - \lambda I_{\hat{\epsilon}}\,, \tag{5.70}$$

$$(\text{new}):\ \frac{\langle\text{new}|\sigma^\dagger(z_1,\bar{z}_1)\sigma(0,0)|0,0;\alpha_0\rangle}{\langle\text{new}|0,0;\alpha_0\rangle} = I_{\hat{I}} + \frac{1}{\lambda^3}I_{\hat{\epsilon}}\,. \tag{5.71}$$

We have seen that the 2-point functions are determined completely, and it is easy to verify that the results have reasonable near-boundary behaviours. We have checked (by computer) that the six conformal blocks I_σ, I_ψ, I_I, $I_{\hat{I}}$, I_ϵ, $I_{\hat{\epsilon}}$ all satisfy the 6th order differential equation obtained from (5.34), and also that their Wronskian is non-trivial. Thus these conformal blocks are indeed six independent solutions of the differential equation.

6. Discussion

The purpose of this paper has been to describe the extension of the Coulomb gas formulation of boundary CFTs to models with a higher chiral algebra. We have shown in detail how this works for A_2 to yield exactly the conformal boundary states that are expected on the basis of general arguments for the three state Potts model. At the level of a single boundary (ie a disk topology) the correlation functions that we have computed are all consistent with our understanding of the physics of the model. The σ two point amplitudes lead naturally to the two extra conformal blocks that are expected over and above those associated in the ordinary (ie as in rational minimal models) way with the primary fields. This is very satisfactory; in this formulation we can calculate an integral representation for essentially any correlation function of the CFT in the disk topology, just as the original Coulomb gas method can for the bulk theory.

There are some questions that are not settled. Firstly the work reported here is not a complete test of the formulation at higher topology. The expectation of the identity operator in the cylinder topology is given correctly (ie as the cylinder amplitudes) but we have not calculated other correlation functions and the constants κ_{wMN} in (4.20) have not been completely determined (at least up to trivial ambiguities). This is all intimately related to the BRST formulation of the Felder complex for the W_3 algebra and there seem to be some gaps in what is known about this; for example, so far as we know, there is no proof of the statement (3.17) in the literature. Secondly it is clear that the general structure of this formulation can be extended without much difficulty to higher W algebras and can be used as a means of classifying possible conformal invariant boundary states which do not preserve the higher symmetry; whether all such states can be found this way is an open question.

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A. W conserving screening operators

Note that W invariance of Q_α is equivalent to the statement

$$W(z)V_\alpha(w) = \partial_w(\ldots) + \text{regular terms}\,. \tag{A.1}$$

In turn this can be checked by considering whether it is the case that

$$\mathcal{D}_3 : e^{i\alpha\cdot\phi(w)} := \partial_w(\ldots) + \text{regular terms}\,. \tag{A.2}$$

If so then, since we know that the Virasoro generators commute with Q_α, it would follow that so does W. It is a straightforward but tedious exercise to use Wick's theorem to compute the singular parts of the left hand side of (A.2) for a general α,

$$\alpha = \alpha_1 e_1 + \alpha_2 e_2\,. \tag{A.3}$$

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After repeatedly discarding total derivatives with respect to w we are left with

$$
\begin{aligned}
: \frac{(2i\alpha_0)^{-3}}{z-w}\Bigg(& -2\alpha_0\alpha_1 e_1\cdot\partial^2\phi - \frac{1}{2}\alpha_1(4\alpha_0-\alpha_2)(2\alpha_0+\alpha_2-\alpha_1)\alpha\cdot\partial^2\phi + \\
& + i\Big(\alpha_2 h_1\cdot\partial\phi\, e_2\cdot\partial\phi + \alpha_1 h_3\cdot\partial\phi\, e_1\cdot\partial\phi + \\
& \quad + \alpha\cdot\partial\phi\Big((2\alpha_0\alpha_2-4\alpha_0\alpha_1-\alpha_2^2+\alpha_2\alpha_1)h_1\cdot\partial\phi + \alpha_1(\alpha_2-\alpha_1)h_3\cdot\partial\phi - \\
& \qquad - \alpha_1\alpha_2 h_2\cdot\partial\phi - \frac{\alpha_1}{2}(4\alpha_0-\alpha_2)(2\alpha_0+\alpha_2-\alpha_1)\alpha\cdot\partial\phi\Big)\Big)\Bigg) : V_\alpha(w)\,.
\end{aligned}
\tag{A.4}
$$

In order for (A.2) to hold this must be zero. The linear terms in ϕ give

$$
\begin{aligned}
-2\alpha_0\alpha_1 - \frac{1}{2}\alpha_1^2(4\alpha_0-\alpha_2)(2\alpha_0+\alpha_2-\alpha_1) &= 0\,, \\
-\frac{1}{2}\alpha_1\alpha_2(4\alpha_0-\alpha_2)(2\alpha_0+\alpha_2-\alpha_1) &= 0\,.
\end{aligned}
\tag{A.5}
$$

The first solution is $\alpha_2 = 0$, $\alpha_1 = \alpha_\pm$ where

$$
\alpha_\pm^2 - 2\alpha_0\alpha_\pm = 1\,. \tag{A.6}
$$

The second solution is $\alpha_1 = 0$ and we must then examine the quadratic terms in ϕ which show that $\alpha_2 = \alpha_\pm$. There are no other solutions so we see that the *only* W conserving screening operators are

$$
\begin{aligned}
Q_\pm^{(1)} &= \oint dz : e^{i\alpha_\pm e_1\cdot\phi(z)} :\,, \\
Q_\pm^{(2)} &= \oint dz : e^{i\alpha_\pm e_2\cdot\phi(z)} :\,,
\end{aligned}
\tag{A.7}
$$

and that all the others violate the W symmetry.

B. Θ function identities

The equivalence of the cylinder amplitudes (4.22) and (4.24) is not entirely straightforward to prove. We show here one way of doing it for the mixed amplitude in the Identity representation case. The techniques we use are described for example in the book [19].

Using the standard results for the Virasoro characters in (4.24) gives us

$$
\begin{aligned}
R &= \chi_{1,1} - \chi_{4,1} \\
&= \frac{q^{-\frac{1}{30}}}{\prod_{k>0}(1-q^k)} \sum_{n\in\mathbb{Z}} q^{30n^2+n} - q^{30n^2+11n+1} - q^{30n^2+19n+3} + q^{30n^2+29n+7}\,.
\end{aligned}
\tag{B.1}
$$

On the other hand from (4.22) we expect this to be equal to

$$
L = \frac{q^{-\frac{1}{30}}}{\prod_{k>0}(1-q^{2k})} \sum_{n\in\mathbb{Z}} q^{20n^2+2n} - q^{20n^2+18n+4}\,. \tag{B.2}
$$

First note that L can be rewritten

$$L = \frac{q^{-\frac{1}{30}}}{\prod_{k>0}(1-q^{2k})} \sum_{n\in\mathbb{Z}} (-1)^n q^{5n^2-n} \tag{B.3}$$

and then consider

$$\begin{aligned} L' &= q^{\frac{1}{30}} \prod_{k>0} (1-q^k)^2 L \\ &= \prod_{k>0} (1-q^{2k})(1-q^{2k-1})^2 \sum_{n\in\mathbb{Z}} (-1)^n q^{5n^2-n} \\ &= \sum_{m,n\in\mathbb{Z}} (-1)^m q^{m^2} (-1)^n q^{5n^2-n}\,, \end{aligned} \tag{B.4}$$

where in the last step we have used the Jacobi triple product formula. Similarly we find that

$$\begin{aligned} R' &= q^{\frac{1}{30}} \prod_{k>0} (1-q^k)^2 R \\ &= \sum_{m,n\in\mathbb{Z}} (-1)^m q^{\frac{3}{2}m^2-\frac{1}{2}m} \left(q^{30n^2+n} - q^{30n^2+11n+1} - q^{30n^2+19n+3} + q^{30n^2+29n+7}\right) \\ &= \sum_{m,n\in\mathbb{Z}} (q^{6m^2-m} - q^{6m^2+5m+1}) \times \\ &\qquad \times \left(q^{30n^2+n} - q^{30n^2+11n+1} - q^{30n^2+19n+3} + q^{30n^2+29n+7}\right). \end{aligned} \tag{B.5}$$

We wish to prove that $Q \equiv L' - R' = 0$.

Note that writing

$$m = k - l\,, \qquad n = 5k + l + p\,, \tag{B.6}$$

and summing over $k, l \in \mathbb{Z}$ and $p = 0, 1, 2, 3, 4, 5$ is equivalent to summing over $m, n \in \mathbb{Z}$. Hence

$$L' = \sum_{k,l\in\mathbb{Z}} \sum_{p=0}^{5} (-1)^p q^{30k^2+6l^2+2p(5k+l)-k+l+p^2}\,, \tag{B.7}$$

and so using (B.5) and (B.7) we find

$$\begin{aligned} Q &= \sum_{l,k\in\mathbb{Z}} q^{6l^2+l} q^{30k^2+19k+3} - \sum_{l,k\in\mathbb{Z}} q^{6l^2+3l} q^{30k^2+9k+1} + \\ &\quad + \sum_{l,k\in\mathbb{Z}} q^{6l^2+5l+l} q^{30k^2+k} - \sum_{l,k\in\mathbb{Z}} q^{6l^2+7l+2} q^{30k^2+11k+1} + \\ &\quad + \sum_{l,k\in\mathbb{Z}} q^{6l^2+9l} q^{30k^2+39k+16} - \sum_{l,k\in\mathbb{Z}} q^{6l^2+l} q^{30k^2+29k+7} \\ &= \sum_{k,l\in\mathbb{Z}} \sum_{p=0}^{5} (-1)^p q^{30k^2+6l^2+(19-10p)k+(2p+1)l+p^2-3p+3}\,. \end{aligned} \tag{B.8}$$

Now change the sign of k in (B.8) and make the change of variables (B.6) in reverse to find that

$$Q = \sum_{m,n\in\mathbb{Z}} (-1)^{m+n} q^{5n^2-4n+(m-\frac{3}{2})^2+\frac{3}{4}}\,. \tag{B.9}$$

We see immediately that the power of q in Q is invariant under $m \to 3-m$ but the phase factor changes sign; hence $Q = 0$.

References

[1] V.S. Dotsenko and V.A. Fateev, *Conformal algebra and multipoint correlation functions in 2d statistical models*, *Nucl. Phys.* **B 240** (1984) 312.

[2] V.S. Dotsenko and V.A. Fateev, *Four point correlation functions and the operator algebra in the two-dimensional conformal invariant theories with the central charge $c \leq 1$*, *Nucl. Phys.* **B 251** (1985) 691.

[3] J.L. Cardy, *Conformal invariance and surface critical behavior*, *Nucl. Phys.* **B 240** (1984) 514.

[4] S. Kawai, *Coulomb-gas approach for boundary conformal field theory*, *Nucl. Phys.* **B 630** (2002) 203 [hep-th/0201146].

[5] S. Kawai, *Free-field realisation of boundary states and boundary correlation functions of minimal models*, *J. Phys.* **A 36** (2003) 6547 [hep-th/0210032].

[6] V.A. Fateev and A.B. Zamolodchikov, *Conformal quantum field theory models in two-dimensions having Z_3 symmetry*, *Nucl. Phys.* **B 280** (1987) 644.

[7] J.L. Cardy, *Boundary conditions, fusion rules and the Verlinde formula*, *Nucl. Phys.* **B 324** (1989) 581.

[8] I. Affleck, M. Oshikawa and H. Saleur, *Boundary critical phenomena in the three-state Potts model*, cond-mat/9804117.

[9] J. Fuchs and C. Schweigert, *Completeness of boundary conditions for the critical three-state Potts model*, *Phys. Lett.* **B 441** (1998) 141 [hep-th/9806121].

[10] R.E. Behrend, P.A. Pearce and J.-B. Zuber, *Integrable boundaries, conformal boundary conditions and A-D-E fusion rules*, *J. Phys.* **A 31** (1998) L763 [hep-th/9807142].

[11] G. Felder, *BRST approach to minimal methods*, *Nucl. Phys.* **B 317** (1989) 215.

[12] A. Bilal, *Introduction to W algebras*, presented at the *Spring School on String Theory and Quantum Gravity*, Trieste, Italy, Apr. 15–23, 1991.

[13] P. Bouwknegt and K. Schoutens, *W symmetry in conformal field theory*, *Phys. Rept.* **223** (1993) 183 [hep-th/9210010].

[14] P. Bouwknegt, J.G. McCarthy and K. Pilch, *Quantum group structure in the Fock space resolutions of* SL(N) *representations*, *Commun. Math. Phys.* **131** (1990) 125.

[15] S. Mizoguchi and T. Nakatsu, *BRST structure of the W_3 minimal model*, *Prog. Theor. Phys.* **87** (1992) 727.

[16] P. Di Francesco, P. Mathieu, and D. Senechal, *Conformal field theory*, Springer, New York 1997.

[17] J.L. Cardy and D.C. Lewellen, *Bulk and boundary operators in conformal field theory*, *Phys. Lett.* **B 259** (1991) 274.

[18] S. Mizoguchi, *The structure of representation of the W_3 algebra*, *Int. J. Mod. Phys.* **A 6** (1991) 133.

[19] H.F. Baker, *Abelian functions*, Cambridge University Press, Cambridge 1897, reissued 1995.